

A damage accumulation model for destruction of tall buildings due to impact loads

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Abstract

Traditional modeling of damage and destruction of tall buildings that are subjected to impact loads does not produce satisfactory results due to some principal difficulties. An alternative energy approach is introduced in this paper for modeling damage accumulation and collapse processes in an effective porous medium representing a building structure. This theory describes several effects, such as the threshold stress, at which damage accumulation begins, and rheological instability related to the macro-fracture of a material and to the strength associated with tension, shear, and compression. The propagation of stress waves caused by a compressive impulse applied to the top of a building is investigated, and various types of destruction and wave transformation are revealed. The formation of a macro-fracture wave and a delay of the building collapse are studied. The influence of material parameters of the effective medium on the structural strength is discussed.

Introduction

Tall buildings are an integral part of a big city infrastructure. High-intensity actions exerted on tall buildings, such as an airplane crash, explosion, or fire, caused by a terrorist attack or other extreme events, may result in partial or complete destruction of such constructions and lead to significant material and human losses. Vulnerability of tall buildings and means for improving their survivability and minimizing losses are not well studied; and therefore, they represent a challenging task in applied mechanics (Bazant & Zhou 2001).

Dynamic damage models have been applied to the problems with high-rate loading of compact materials, such as metals, minerals, and concretes (e.g. Ning et al. 2003, Zhang and Hao 2003). However, acceptable methodologies addressing the damage and collapse of buildings do not exist, because of both technical and principal problems. The most important of these problems are the following:

- The three-dimensional form of the problem of deformation and damage of a construction with numerous structural elements, such as beams, plates, and membranes. This is linked to nonsymmetrical loads, as well as to the necessity of considering multi-dimensional connections. The exact modeling of the wave redistribution in these connections is not possible within approximate theories of beams and membranes. A complete system of equations must be used for accurate calculations.
- The drastic difference among characteristic times of dynamic impacts, thermal exposure, and viscoplastic bending of columns during building collapse. The process of deformation and destruction includes both fast dynamic phenomena and a phase of quasi-static evolution under high loads. A quasi-static phase may again turn into a fast dynamic process.
- The necessity to account for initial stresses in a structure, which possesses a vast amount of elastic energy. This energy is released following a propagation of the destruction wave initiated by a combination of effects of impact, explosion, and fire.
- The absence of deterministic models for damage of initially loaded objects that would accurately establish a form and places under destruction (fragmentation).
- The necessity to carry out calculations on large time intervals due to significant difference in characteristic times. Such time intervals exceed by orders of magnitude a time of the acoustic wave propagation along the building.

Therefore, a traditional usage of finite difference and finite element methods for calculating dynamic and quasi-static processes is not promising due to error accumulation and loss of approximation. Computational codes, which are commonly used at the present time for solid mechanics problems of deforming bodies, will inevitably produce a distorted solution and give inaccurate prediction of consequences of the combined actions of impact, explosion, and fire.

What alternative means can be developed to describe the damage and collapse of a tall building? One of the possible approaches is the calculation based on a continuum model of a building, which is considered as an *effective continuum medium* with several specific characteristics:

- The porosity coefficient is close to one. This coefficient is defined as one minus volumetric fraction of structural elements in the total volume of a building.
- The elastic response to low loads and the irreversible damage accumulation under high loads.

- The presence of both the shear and volumetric damage (contraction) of a structure under high loads.
- The energy release upon destruction of the initially stressed building.
- The fast growth and localization of damage in the states approaching macro-scale destruction.

To analyze the behavior of a medium with features described above, the energetic model of *continuum (scattered) damage* is applied in this study, e.g. (Kondaurov & Kutlyarova 2000, Kondaurov & Fortov 2002) and for other references (Holzapfel 2001). The development of this model is based on a new concept of the effective surface energy of a micro-defect ensemble and the latent energy of structural transformations. This energy method is related to Griffith's approach in mechanics of an isolated fracture (Griffith 1920, Cherepanov 1974, Freund 1998). The central point of this approach is a local balance between the elastic energy of a deformed body and the surface energy of micro-cracks and micro-pores. The strain and strength characteristics of the continuum can be interrelated accounting for the irreversible increase of the surface defect energy and assuming a reduction in the potential energy of deformation, which is released upon the formation and growth of defects due to partial unloading in the material surrounding these defects. From the thermodynamic point of view, this means that the description of evolution can be related to the accounting for distributed sources in the energy conservation law. These sources have a surface origin and are neither thermal nor mechanical forms of energy.

Kinetics of scattered destruction is based on the assumption that the rate of damage is proportional to derivative of the total energy on the damage variable, where the total energy is the sum of the elastic potential energy and the effective surface energy. This kinetic equation not only provides the non-negativity of dissipation in any process of damage accumulation but also allows us to define a scalar measure for the "dynamic overload" in any multi-axial stress-strain state.

Following (Rudnicki & Rice 1975, Kondaurov 1991), Hadamard condition (Courant 1962, Trusdell 1972) is used as the elementary strength criterion that assumes the real velocities of non-stationary characteristics of dynamic equations for the considered material. Violation of this condition results in the ill-posed boundary problem. Therefore, this restriction on permissible strains and damage plays a role of the threshold that cannot be exceeded. It should be noted that this strength criterion follows from the considered equations and is directly related to the mechanical properties of a medium.

1. Basic equations

Only small deformations will be considered in mathematical modeling of an effective continuum with the coefficient of porosity close to one. The influence of temperature will be neglected. The effective medium is assumed homogeneous and isotropic; although more advance approach would treat this medium as having orthotropic symmetry or transversal isotropy.

Let \mathbf{u} be the displacement, $\mathbf{v} = \dot{\mathbf{u}}$ is the velocity of a material particle, and $\mathbf{e} = \frac{1}{2}(\nabla \otimes \mathbf{u} + (\nabla \otimes \mathbf{u})^T)$ is the symmetric tensor of small deformations. Here and further, the dot over a symbol designates the time derivative, ∇ is the gradient on spatial coordinate \mathbf{x} .

Let the current state of the medium element be assigned deformation \mathbf{e} and a damage scalar $\omega > 0$. The medium response is defined by three relationships

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}(\mathbf{e}, \omega), \quad u = u(\mathbf{e}, \omega), \quad u_f = u_f(\omega), \quad (1.1)$$

which define the symmetric stress tensor $\boldsymbol{\sigma}$ and elastic potential u as isotropic functions of deformation and damage. The effective surface energy u_f is the function of damage only. For the elastic potential and the surface energy, the following expressions are taken

$$\rho u(\mathbf{e}, \omega) = \frac{1}{2} K I_1^2(\mathbf{e}) + \mu J^2(\mathbf{e}) - \alpha_p (I_1) \omega I_1(\mathbf{e}) - \alpha_s \omega J(\mathbf{e}), \quad (1.2)$$

$$\rho u_f(\omega) = \rho u_f^0 + \gamma \omega + \frac{1}{2} \beta \omega^2, \quad (1.3)$$

$$\rho, K, \mu, \alpha_s = \text{const} > 0,$$

where ρ is the material density, which can be considered constant and equal to the density in the initial state since deformations are small, $I_1(\mathbf{e}) = \mathbf{e} : \mathbf{I}$, $J(\mathbf{e}) = (\mathbf{e}' : \mathbf{e}')^{1/2}$, $\mathbf{e}' = \mathbf{e} - \frac{1}{3} I_1 \mathbf{I}$ is the deviator of elastic deformation tensor. Coefficients in equations (1.2) and (1.3) – $K, \mu, \alpha_s, \beta, u_f^0, \gamma$ – are the medium parameters which are assumed to be dependent only on the properties of the matrix composed of structural elements of a building and on the initial porosity.

The elastic potential (1.2) is the decomposition in the Taylor series up to second order on deformation and damage. The first two terms in eq. (1.2) correspond to the energy of an isotropic, linearly elastic body obeying Hooke's law. The last two terms describe a decrease of the elastic energy due to damage accumulation (note minus signs). Coefficients α_p and α_s characterize a reduction of energy related to volumetric and shear strains, respectively, in the presence of damage. The terms of the first order in (1.2) do not exist, because the reference configuration is the stress-free configuration. The term of the zero order is included into effective surface energy $\rho u_f(\omega)$. Function $\alpha_p(I_1)$ is considered to be sign alternating, which is related to the reduction of the elastic energy in the processes of damage accumulation, at both high-intensive tension ($I_1(\mathbf{e}) > 0$) and compression ($I_1(\mathbf{e}) < 0$).

The simplest way to assign numerical values to elastic coefficients K and μ is to use elastic modulus of structural elements (e.g. concrete) multiplied by the volumetric fraction of the structural elements in the total volume of a building. More accurate approximations should be based on methods for determining the effective stiffness, which are developed in mechanics of composite materials (Rabotnov 1979, Christensen 1979). Especially attractive seems the method of asymptotic averaging of periodic and random structures (Bahvalov 1974, Bahvalov 1975, Berdichevsky 1983).

Conservation laws for medium under consideration include the relation between deformation and velocity, momentum equation (\mathbf{g} is the gravitational acceleration), and local energy balance:

$$\dot{\mathbf{e}} = \frac{1}{2}(\nabla \otimes \mathbf{v} + (\nabla \otimes \mathbf{v})^T), \quad (1.4)$$

$$\rho \dot{\mathbf{v}} = \nabla \cdot \boldsymbol{\sigma} + \rho \mathbf{g}, \quad (1.5)$$

$$\rho \dot{u}(\mathbf{e}, \omega) = \boldsymbol{\sigma} : \dot{\mathbf{e}} - \rho \dot{u}_f(\omega) - \delta. \quad (1.6)$$

Distributed energy sinks are taken into account in equation (1.6). The function $\rho \dot{u}_f(\omega)$ is associated with the energy expense for increasing effective surface energy ρu_f accompanying the formation of micro-fractures, i.e. at the increase of damage variable ω . Energetic damage δ in the isothermal process is exactly equal to the heat produced in deforming material and absorbed by the environment.

In the considered medium, the time derivative of its damage variable is determined by the *kinetic equation* as follows

$$\dot{\omega} = \frac{1}{\tau} \Phi \left(-\frac{1}{\beta} \frac{\partial \rho(u(\mathbf{e}, \omega) + u_f(\omega))}{\partial \omega} \right),$$

where $\tau > 0$ is the *stress relaxation time* and the argument of kinetic function $\Phi(\mathbf{e}, \omega)$ is the difference between released elastic energy and absorbed surface energy. In the following, only the simplest form will be used

$$\dot{\omega} = \frac{1}{\tau \beta} z H(z) H(\omega), \quad z = \alpha_p(I_1)I_1 + \alpha_s J - \gamma - \beta \omega, \quad (1.7)$$

where $H(x)$ is the Heaviside function. From equation (1.6), it follows that the damage increases under the condition of *active loading*

$$\alpha_p(I_1)I_1 + \alpha_s J - \gamma - \beta \omega > 0, \quad \omega \geq 0.$$

If this condition is not fulfilled then the damage variable does not change. Dissipation in the process of active loading is always positive for the chosen damage kinetics

$$\delta = -\frac{\partial \rho(u + u_f)}{\partial \omega} \dot{\omega} = \frac{1}{\tau \beta} \left[\frac{\partial \rho(u(\mathbf{e}, \omega) + u_f(\omega))}{\partial \omega} \right]^2.$$

Since the strain rate $\dot{\mathbf{e}}$ is arbitrary then it follows from (1.6) that the stress tensor is related to the elastic potential

$$\boldsymbol{\sigma} = \rho \frac{\partial u(\mathbf{e}, \omega)}{\partial \mathbf{e}} = (KI_1(\mathbf{e}) - \tilde{\alpha}_p(I_1)\omega) \mathbf{I} + \left(2\mu - \frac{\alpha_s \omega}{J(\mathbf{e})} \right) \mathbf{e}', \quad \tilde{\alpha}_p(I_1) \equiv \alpha_p(I_1) + I_1 \alpha_p'(I_1). \quad (1.8)$$

Besides the positively defined dissipation in the processes of damage accumulation, there are other reasons for introducing (1.7) into the model:

- In a slow process ($\dot{\omega} \rightarrow 0$), equation (1.7) leads to the model of a damaged body based on the local balance between the elastic energy and the effective surface energy (Kondaurov 1988, Kondaurov & Fortov 2002).

- Expression $-\rho\partial(u + u_f)/\partial\omega$, which is equal to the difference of the rates of the elastic and surface energy, is a natural scalar measure of the dynamic overload, which is applicable for different types of a stress-strain state.
- Equation (1.7) allows us to describe in a similar way the qualitative specific features of the continuous fracture, in particularly, the appearance of the threshold deformation levels when a damage accumulation starts, the possibility of damage due to tension, shear, and compression, and the absence of damage evolution upon unloading below a certain threshold.

The specific properties of an initially porous media discussed above leads to subdivision of semi-plane $(I_1(\mathbf{e}), J(\mathbf{e}))$ of strain invariants into two areas shown in Fig. 1. Curve $J = f(I_1)$ restricts area 1 of the undamaged material and has a considerable asymmetry with respect to the J -axis, which is related to the essential difference of the strength properties in compression and extension. The area 2 corresponds to the damaged material.

If the boundary of the elastic area $J = f(I_1)$ is known, then condition $\alpha_p(I_1)I_1 + \alpha_s J - \gamma = 0$ of the beginning of damage accumulation provides the relation between function $\alpha_p(I_1)$, parameter γ , and the boundary of the elastic area:

$$\gamma = \alpha_s f(0), \quad \alpha_p(I_1) = \alpha_s (f(0) - f(I_1)) / I_1. \quad (1.9)$$

For convex function $f(I_1)$, the value of $\alpha_p(I_1)$ is positive in the interval $I_1 \in (I_1^*, I_1^+)$ and negative in $I_1 \in (I_1^-, I_1^*)$. Here, I_1^\pm are the threshold values of the volumetric deformation of stretch and compression when the damage accumulation starts, $I_1^* < 0$ is the volumetric deformation of compression such that $f(I_1^*) = f(0)$. The dependence of invariant I_1 of function $\alpha_p(I_1)$ and a sign alternation of this function strongly affect reology of the material and are caused by the initial porosity of the medium.

The specifics of curve $J = f(I_1)$ that separates the areas of elastic and inelastic behavior of initially porous or initially fractured media are discussed in the literature and are used in calculating short- and long-term strength of concretes, geomaterials, and metallic powders (Fleck & Kuhn 1992, Aubertin & Simon 2000, Lundgren & Magnusson 2001). Figure 2 schematically shows a generalized strength criterion of such media, where Γ is the continuity parameter, reduction of which characterizes the increase of initial porosity.

In this paper, such curves are applied for the phenomenological modeling of the response of an effective initially stressed material of a tall building. It should be noted that the transition of a deformed state across this boundary does not mean macro-fracture of the material but the beginning of the damage evolution. Macro-fracture starts later when reological instability appears. A family of such states cannot be depicted on the plane of current deformations $(I_1(\mathbf{e}), J(\mathbf{e}))$ because these states are the functionals defined on the entire strain history. These questions are considered in details in the next section.

From now on, only a piece-wise linear function for the boundary of the undamaged material states will be used, similar to that proposed previously (Kondaurov & Kutlyarova 2000, Kondaurov & Fortov 2002)

$$J(I_1) = \begin{cases} J_0(1 - I_1/I_1^+), & I_1^* \leq I_1 \leq I_1^+ \\ J_1(1 - I_1/I_1^-), & I_1^- \leq I_1 \leq I_1^* \end{cases}, \quad J_1 = J_0 \frac{1 - I_1^*/I_1^+}{1 - I_1^*/I_1^-}, \quad (1.10)$$

where $J_0 = J(0)$ is the intersection point of the right part of the boundary with axis J , and J_1 is the intersection point of the extended left boundary with the ordinate axis. Shown in Fig. 3, the piece-wise linear boundary allows us to significantly simplify mathematical derivation but retain all qualitative properties of the behavior of an initially porous material. In this case, the coefficient appearing in equation (1.8) is

$$\tilde{\alpha}_p(I_1) = \begin{cases} \alpha_p^+ \equiv \alpha_s J_0 / I_1^+ > 0, & I_1^* \leq I_1 \leq I_1^+ \\ \alpha_p^- \equiv \alpha_s J_1 / I_1^- < 0, & I_1^- \leq I_1 \leq I_1^* \end{cases}. \quad (1.11)$$

The relation (1.8) between stress, strain and damage becomes much simpler than that in the case of continuous function $\alpha_p(I_1)$.

2. Characteristics of the system of equations of the medium undergoing damage

The model constructed above will be applied here for investigating the long-term strength criterion, corresponding to the loss of the material ability to carry loads. Following some previous studies (Kukudjanov 1977,

Rudnicki & Rice 1975, Kondaurov 2001) we will formulate the conditions when characteristic surfaces of the dynamic system propagate with real velocities. As will be seen later, the velocities of characteristic surfaces (weak discontinuity, acoustic disturbances) in the studied medium depend on the current strain and damage. At degeneration, when the velocity of one of the unsteady characteristic surfaces becomes zero, a weak discontinuity (where the solution derivatives experience a jump) transforms into a strong shock (with discontinuity in the solution itself). In particular, the deformation tensor experiences a jump at such a surface, i.e. the *strain localization* occurs. Formation of these surfaces is accompanied by the loss of Hadamard condition (Courant 1962, Trusdell 1972), which is a necessary condition of hyperbolicity of the dynamic system. The fact that initial and boundary problems for the considered systems become ill-posed means that studying subsequent deformations in a medium cannot be carried out within the stated model. Hence, the limit for possible deformations and accumulated damage, associated with the real type of characteristics velocities, plays a role of the *strength criterion*, which cannot be exceeded for the considered medium and which depends on the properties of the material.

Dynamics of an initially porous medium in the adiabatic approximation (heat flux and density of distributed heat sources are zero) is described in the active process ($\omega > 0$, $\dot{\omega} > 0$) by the system of equations (1.4), (1.5), and (1.9):

$$\begin{aligned} \rho \dot{\mathbf{v}} - \nabla \cdot \boldsymbol{\sigma}(\mathbf{e}, \omega) &= \rho \mathbf{g}, & 2\dot{\mathbf{e}} - \nabla \otimes \mathbf{v} - (\nabla \otimes \mathbf{v})^T &= 0 \\ \dot{\omega} + \frac{1}{\tau} \omega &= \frac{1}{\tau \beta} (\alpha_p (I_1) I_1 + \alpha_s J - \gamma) \end{aligned} \quad (2.1)$$

where the deformation tensor is defined by (1.8).

Let \mathbf{I} be a unity tensor of the second rank, δ_{ik} is the Kronecker symbol, $\mathbf{N}(\mathbf{e}) = \mathbf{e}' / J$ is the normalized deviator of the deformation tensor such that $\mathbf{N}(\mathbf{e}) : \mathbf{N}(\mathbf{e}) = 1$, $\mathbf{N}(\mathbf{e}) : \mathbf{I} = 0$. Let $w(\mathbf{x}, t) = 0$ be an equation for the weak discontinuity surface of the solution of (2.1). Then the velocity of propagation and the normal to that surface are defined as follows

$$c = -\frac{1}{|\nabla w|} \frac{\partial w(\mathbf{x}, t)}{\partial t}, \quad \mathbf{n} = \frac{\nabla w}{|\nabla w|}$$

The material on both sides of the surface $w(\mathbf{x}, t) = 0$ is assumed to be in active loading; therefore, solution $(\mathbf{v}, \mathbf{e}, \varphi)$ and coefficients, as well as the right-hand sides of equations (2.1), are continuous. The surface $w(\mathbf{x}, t) = 0$ will be the characteristic surface of the system of partial-differential equations (2.1) if a non-unique continuation of the solution through this surface is possible (Courant 1962).

The propagation velocities of unsteady characteristic surfaces ($c \neq 0$) of the system (2.1) are determined by the characteristic equation $\det(\rho c^2 \mathbf{I} - \mathbf{A}) = 0$, where the *acoustic tensor* is

$$\begin{aligned} \mathbf{A}(\mathbf{e}, \omega, \mathbf{n}) &= M \mathbf{I} + \Lambda \mathbf{n} \otimes \mathbf{n} + \xi \mathbf{m} \otimes \mathbf{m}, & (2.2) \\ M(\xi) &= \mu - \frac{\xi}{2}, & \Lambda(\xi) &= \lambda + \mu - \frac{\xi}{6}, & \xi(\mathbf{e}, \omega) &= \frac{\alpha_s \omega}{J(\mathbf{e})}, & \mathbf{m} &= \mathbf{N}(\mathbf{e}) \cdot \mathbf{n}. \end{aligned}$$

Eigenvalues of the acoustic tensor \mathbf{A} , i.e. the propagation velocities of the weak discontinuity surfaces, are determined by formula

$$\rho c_{1,2}^2(\mathbf{e}, \omega, \mathbf{n}) = M + P \pm (P^2 - Q)^{1/2}, \quad \rho c_3^2(\mathbf{e}, \omega) = M, \quad (2.3)$$

where

$$\begin{aligned} 2P(\mathbf{e}, \omega, \mathbf{n}) &= \Lambda + \xi \mathbf{m} \cdot \mathbf{m} = \lambda + \mu + \xi (\mathbf{n} \cdot \mathbf{N}^2(\mathbf{e}) \cdot \mathbf{n} - 1/6) \\ Q(\mathbf{e}, \omega, \mathbf{n}) &= \xi \Lambda (\mathbf{m} \cdot \mathbf{m} - (\mathbf{m} \cdot \mathbf{n})^2) = \xi \Lambda (\mathbf{n} \cdot \mathbf{N}^2(\mathbf{e}) \cdot \mathbf{n} - (\mathbf{n} \cdot \mathbf{N}(\mathbf{e}) \cdot \mathbf{n})^2) \end{aligned} \quad (2.4)$$

The dependence of the characteristics velocities in the considered material on the current deformation and damage indicates that at some values of $\xi(\mathbf{e}, \omega) = \alpha_s \omega / J(\mathbf{e})$ and material parameters $\lambda, \mu, \alpha_p, \alpha_s$ there are directions \mathbf{n} for which the velocity of unsteady characteristics becomes zero. In this case, the positively determined acoustic tensor \mathbf{A} *degenerates*; and several mechanical effects accompany this degeneration.

Following previous studies (Rudnicki & Rice 1975, Kondaurov 1991) let us call the state $(\mathbf{e}^{cr}, \varphi^{cr})$ of a material element to be *reologically unstable* if there is a direction $\mathbf{n}_{cr}(\mathbf{e}^{cr}, \varphi^{cr})$ along which the characteristic surface velocity is zero, i.e. $c(\mathbf{e}^{cr}, \varphi^{cr}, \mathbf{n}_{cr}) = 0$. The reological instability shows one's worth formation of the strain

localization surfaces with a certain well-defined orientation $\mathbf{n}_{cr}(\mathbf{e}^{cr}(\mathbf{x}, t), \varphi^{cr}(\mathbf{x}, t))$ of these elements. In fact, vector $\mathbf{V} \equiv [\partial \mathbf{v} / \partial n]$ of the jump of the normal derivative of the mass velocity, defined by $(\rho c^2 \mathbf{I} - \mathbf{A}) \cdot \mathbf{V} = 0$, approached to a non-zero value $\mathbf{V}_0 \neq 0$ as $c \rightarrow 0$. From the second equation (2.1), one can see that at $\mathbf{V}_0 \neq 0$ the amplitude $\mathbf{E} \equiv [\partial \mathbf{e} / \partial n]$ of the jump of the normal derivative of the deformation tensor approaches infinity as $c \rightarrow 0$. This means that a weak shock becomes a strong shock, through which the velocity is continuous but deformation is discontinuous. The surface of such a shock is called the surface of *strain localization*.

From the second equation (2.3), it follows that the degeneration on velocity $\rho c_3^2(\mathbf{e}, \omega) = \mu - \frac{1}{2} \xi$ occurs in the states that satisfy the condition

$$\xi_{cr} \equiv \alpha_s \omega^{cr} / J(\mathbf{e}^{cr}) = 2\mu. \quad (2.5)$$

The orientation of the localization surface, defined by vector \mathbf{n}_{cr} , is arbitrary in this case because of the independence of condition (2.5) on the direction of the characteristic surface propagation. The unity vector \mathbf{n} that provides the extremum for the value $\rho c_2^2(\mathbf{e}, \omega, \mathbf{n})$, which is always less than ρc_1^2 , is determined by the equation

$$(\mathbf{I} - \mathbf{n} \otimes \mathbf{n}) \cdot \mathbf{B} \cdot \mathbf{n} = 0, \quad (2.6)$$

where the symmetric second-rank tensor is

$$\mathbf{B} \equiv (\Lambda - S_2) \mathbf{N}^2(\mathbf{e}) - 2\Lambda(\mathbf{n} \cdot \mathbf{N}(\mathbf{e}) \cdot \mathbf{n}) \mathbf{N}(\mathbf{e}), \quad S_2 \equiv \rho c_2^2 - M. \quad (2.7)$$

Equation (2.6) can be written as $\mathbf{B} \cdot \mathbf{n} = (\mathbf{n} \cdot \mathbf{B} \cdot \mathbf{n}) \mathbf{n}$. It follows that \mathbf{n} is an eigenvector of symmetric tensor \mathbf{B} . In view of equation (2.5) the symmetric tensor \mathbf{B} is a second-rank polynomial of normalized deviator $\mathbf{N}(\mathbf{e})$ with coefficients dependent on parameter ξ and invariants $\mathbf{n} \cdot \mathbf{N}(\mathbf{e}) \cdot \mathbf{n}$, $\mathbf{n} \cdot \mathbf{N}^2(\mathbf{e}) \cdot \mathbf{n}$. Therefore, eigenvectors of tensor \mathbf{N} are the eigenvectors of tensor \mathbf{B} . The inverse statement, generally speaking, is incorrect, as some eigenvectors of \mathbf{B} may not be those for \mathbf{N} .

If the orthonormal basis $\mathbf{e}_i, i = 1, 2, 3$ of the coordinate system coincides with the principal axes of tensor \mathbf{N} in the considered point, then

$$\mathbf{B} = B_1 \mathbf{e}_1 \otimes \mathbf{e}_1 + B_2 \mathbf{e}_2 \otimes \mathbf{e}_2 + B_3 \mathbf{e}_3 \otimes \mathbf{e}_3$$

$$\mathbf{B} \cdot \mathbf{n} = B_1 n_1 \mathbf{e}_1 + B_2 n_2 \mathbf{e}_2 + B_3 n_3 \mathbf{e}_3, \quad \mathbf{n} \cdot \mathbf{B} \cdot \mathbf{n} = B_1 n_1^2 + B_2 n_2^2 + B_3 n_3^2$$

Equation (2.6) in this basis corresponds to three scalar equations

$$\begin{aligned} ((B_1 - B_2)n_2^2 + (B_1 - B_3)n_3^2)n_1 &= 0 \\ ((B_2 - B_3)n_3^2 + (B_2 - B_1)n_1^2)n_2 &= 0, \\ ((B_3 - B_1)n_1^2 + (B_3 - B_2)n_2^2)n_3 &= 0 \end{aligned} \quad (2.8)$$

where $B_i, i = 1, 2, 3$ are the eigenvalues of \mathbf{B} . It follows that the system (2.6) at $c_2 = 0$ has the following solutions (Kondaurov & Fortov 2002):

- (1) The normal \mathbf{n} coincides with one of the eigenvectors of tensor $\mathbf{N}(\mathbf{e})$. The reological instability occurs at $\xi_{cr} = 2\mu$ and is exhibited in the form of the surfaces of the *shear deformation discontinuity*, which are perpendicular to the principal axes of tensor $\mathbf{N}(\mathbf{e})$.
- (2) If two eigenvalues of \mathbf{B} are the same, then reological instability appears when $\xi_{cr} > 2\mu$, where ξ_{cr} is the root of the following equation

$$M + P - (P^2 - Q)^{1/2} = 0, \quad (2.9)$$

$$Q = \xi_{cr} \Lambda (N_1 - N_2)^2 n_1^2 n_2^2 > 0, \quad 2P = \Lambda + \xi_{cr} (N_1^2 n_1^2 + N_2^2 n_2^2)$$

$$M(\xi_{cr}) = \mu - \frac{1}{2} \xi_{cr}, \quad \Lambda(\xi_{cr}) = \lambda + \mu - \frac{1}{6} \xi_{cr}$$

One of the principal axes of tensor $\mathbf{N}(\mathbf{e})$ belongs to the discontinuity planes, and for any cyclic combinations 1,2,3 the normal components are found as follows

$$n_1^2 = \frac{1}{2} \left(1 - \frac{M}{\Lambda} \frac{N_3}{(N_1 - N_2)} \right) = \frac{1}{2} \left(1 - \frac{\mu - \xi/2}{\lambda + \mu + \alpha^2/c_e - \xi/6} \frac{N_3}{(N_1 - N_2)} \right) \quad (2.10)$$

$$n_2^2 = 1 - n_1^2, \quad n_3 = 0$$

Both shear and normal components of the deformation tensor experience a discontinuity on these surfaces.

(3) If tensor \mathbf{B} is spherical, which is possible only for uniaxial deformation (for any cyclic combination 1,2,3),

$$\mathbf{e} = e(t) \mathbf{e}_1 \otimes \mathbf{e}_1, \quad \mathbf{N}(\mathbf{e}) = \kappa \sqrt{\frac{2}{3}} \mathbf{e}_1 \otimes \mathbf{e}_1 - \kappa \frac{1}{\sqrt{6}} (\mathbf{e}_2 \otimes \mathbf{e}_2 + \mathbf{e}_3 \otimes \mathbf{e}_3), \quad \kappa = \text{sign}[e(t)],$$

then the extreme normal coincides with the normal to the surface of a circled cone with axis \mathbf{e}_1 . The cone angle 2Ψ is defined by the expression

$$n_1^2 \equiv \sin^2 \Psi = \frac{1}{2} \left(1 + \frac{M}{3\Lambda} \right), \quad n_2^2 + n_3^2 = 1 - n_1^2, \quad \frac{M}{\Lambda} = \frac{\mu - \xi_{cr}/2}{\lambda + \mu - \xi_{cr}/6}. \quad (2.11)$$

where $\xi_{cr} > 2\mu$. As in the second case, both shear and normal components of the strain tensor have a discontinuity on these surfaces.

What form of reological instability realizes in a loading process? We shall suppose the earliest mode of instability corresponds to the earliest moment t_* , when parameter $\xi(\mathbf{e}(t), \omega(t))$ for given deformation trajectory $\mathbf{e}(z)$, $z \leq t_*$ reaches its critical value ξ_{cr} (one of the sound velocities becomes zero). This means that moment $t_* = \min t^{(N)}$, where $t^{(N)}$ is determined by the integral equation

$$\alpha_s \hat{\omega}[\mathbf{e}(s), t^{(N)}]_{s=0}^{s=t^{(N)}} = \xi_{cr}^{(N)} J(\mathbf{e}(t^{(N)})), \quad (2.12)$$

where $\hat{\omega}$ is the integral operator which is a solution of the kinetic equation (1.7) with zero initial conditions, and parameter $\xi_{cr}^{(N)}$ is equal to either 2μ or the root of equation (2.9).

3. The features of material behavior in uniaxial deformation

Consider the *uniaxial compression* with a deformation tensor $\mathbf{e} = e(t) \mathbf{e}_1 \otimes \mathbf{e}_1$, $e(t) < 0$. The first invariant and the deviator of this tensor are defined by formula

$$I_1(\mathbf{e}) = e(t), \quad J = -\sqrt{\frac{2}{3}} e(t), \quad \mathbf{e}' = \frac{1}{3} e(t) (2\mathbf{e}_1 \otimes \mathbf{e}_1 - \mathbf{e}_2 \otimes \mathbf{e}_2 - \mathbf{e}_3 \otimes \mathbf{e}_3). \quad (3.1)$$

The normalized tensor deviator of uniaxial deformation is

$$\mathbf{N}(\mathbf{e}) = \left(-\sqrt{\frac{2}{3}} \mathbf{e}_1 \otimes \mathbf{e}_1 + \frac{1}{\sqrt{6}} \mathbf{e}_2 \otimes \mathbf{e}_2 + \frac{1}{\sqrt{6}} \mathbf{e}_3 \otimes \mathbf{e}_3 \right) = \frac{1}{\sqrt{6}} (\mathbf{I} - 3\mathbf{e}_1 \otimes \mathbf{e}_1). \quad (3.2)$$

On the semi-plane (I_1, J) , this deformation corresponds to the straight line $OABM$ with $J = -\sqrt{2/3} I_1$ (Fig. 3). Two variants of deformation and damage are possible, depending on the material properties. In the first case, when the line $OABM$ crosses the right boundary of the region of undamaged material, i.e.

$J_0(1 - I_1^*/I_1^+) \leq -\sqrt{2/3} I_1^*$, then the process of damage accumulation starts at moderate loads $I_1^* < I_1 < 0$ with the dominant *shear mechanism* of damage. With increasing $|I_1|$, the deformed state transforms into the area of strong compression $I_1^- < I_1 < I_1^*$. The threshold compression $e_{\pm} < 0$, at which the damage starts, is determined by equation $\alpha_p^{\pm} I_1 + \alpha_s J - \gamma = 0$ for the elastic region boundary and it is equal to

$$e_{\pm} = \gamma / a_{\pm}, \quad a_{\pm} = a_p^{\pm} - a_s \sqrt{2/3}. \quad (3.3)$$

If the boundary of the elastic region is such the first loading regime is realized, then the threshold deformation is $e_+ = e_A > I_1^*$; for the second regime $e_- = e_M < I_1^*$.

From equations (1.8) and (1.11), it follows that stress $\sigma_{11}(e)$ in uniaxial deformation, for which equations (3.1)-(3.3) are fulfilled, is

$$\sigma_{11} = Ke - \alpha_p^{\pm} \omega + \frac{4}{3} \mu e + \alpha_s \sqrt{\frac{2}{3}} \omega = \Lambda_0 e - \alpha_{\pm} \omega, \quad \Lambda_0 = K + 4\mu/3 = \lambda + 2\mu. \quad (3.4)$$

The kinetic equation (1.7) then becomes

$$\dot{\omega} = \frac{1}{\tau\beta}(\alpha_{\pm}e - \gamma - \beta\omega) = \frac{1}{\tau}z^{\pm}(\sigma_{11}, \omega), \quad z^{\pm}(\sigma_{11}, \omega) \equiv \frac{\alpha_{\pm}\sigma_{11}}{\beta\Lambda_0} - \left(1 - \frac{\alpha_{\pm}^2}{\beta\Lambda_0}\right)\omega - \frac{\gamma}{\beta}. \quad (3.5)$$

The solution of (3.5) with zero initial conditions can be written in an explicit form. For the first regime of deformation

$$\omega(t) = \frac{1}{\tau\beta} \int_{t_A}^t (\alpha_+ e(s) - \gamma) \exp\left(-\frac{t-s}{\tau}\right) ds, \quad e_A > e(t) > e_B \equiv I_1^* \quad (3.6)$$

$$\omega(t) = \omega(t_B) + \frac{1}{\tau\beta} \int_{t_B}^t (\alpha_- e(s) - \gamma) \exp\left(-\frac{t-s}{\tau}\right) ds, \quad e(t) \leq e_B \equiv I_1^*$$

where t_A is the time moment when the damage begins, i.e. the threshold level is achieved $e_A = \gamma / \alpha_+$; t_B is the time moment such that $e(t_B) = e_B$. For the second regime, there is a single function for $\omega(t)$

$$\omega(t) = \frac{1}{\tau\beta} \int_{t_M}^t (\alpha_- e(s) - \gamma) \exp\left(-\frac{t-s}{\tau}\right) ds, \quad e_M > e(t),$$

where t_M is the time moment associated with achieving the threshold deformation $e_M = \gamma / \alpha_-$.

In the case of a low strain rate, which corresponds to a slow damage rate $\dot{\omega} \rightarrow 0$, the right-hand side of equation (3.5) also approaches zero. Then the damage can be explicitly expressed via deformation e or stress σ_{11}

$$\omega = \beta^{-1}(\alpha_{\pm}e - \gamma), \quad \beta\Lambda_0\omega - \alpha_{\pm}^2\omega = \alpha_{\pm}\sigma_{11} - \gamma\Lambda_0. \quad (3.7)$$

From (3.7) it follows that $\alpha_{\pm} < 0$, since in compression $e < 0$ and damage $\omega > 0$. The expression for the stress σ_{11} at the infinitely small strain rate can be reduced to the piece-wise linear dependence

$$\sigma_{11}(e) = \begin{cases} \Lambda_0 e & e_A < e < 0 \\ \Lambda_f^+(e - e_A) + \sigma_A & e_B < e < e_A \\ \Lambda_f^-(e - e_B) + \sigma_C & e < e_B \end{cases} \quad (3.8)$$

Here, the following symbols are introduced

$$e_A = \gamma / \alpha_+, \quad \sigma_A = \Lambda_0 e_A = \Lambda_0 \gamma / \alpha_+, \quad e_B = I_1^*, \quad \sigma_C = \Lambda_f^- e_B + \gamma \alpha_- / \beta, \quad \Lambda_f^{\pm} = \Lambda_0 - \alpha_{\pm}^2 / \beta$$

Function $\sigma_{11}(e)$ is shown in Fig. 4 by a polyline $OABCD$ for a material with parameters $\lambda = \mu = 0.33$, $\alpha_p^+ = 0.3$, $\alpha_p^- = -0.02$, $\alpha_s = 1.0$, $\gamma = 0.01$, $\beta = 1.0$. There is a discontinuity in the quasi-static stress σ_{11} at $e_B = I_1^*$. Using equation (3.8), which provides $\sigma_B = \Lambda_f^+(e_B - e_A) + \sigma_A$ and $\sigma_C = \Lambda_f^- e_B + \gamma \alpha_- / \beta$, the expression for the jump in stress can be obtained

$$\sigma_C - \sigma_B = \beta^{-1}(\alpha_+ - \alpha_-)\{\alpha_+(e_B - e_A) + \alpha_- e_B\} > 0. \quad (3.9)$$

This jump cannot be negative as follows from the compression condition $e < 0$ and inequalities

$\alpha_+ > \alpha_-$, $\alpha_+ < 0$, $\alpha_- < 0$ (from definition of α_{\pm}). In other words, a release of the compression occurs in achieving point B , which corresponds to the compression I_1^* , i.e. the static diagram of the considered material has a *falling branch*. However, due to the dependence of stress on the strain rate, the instability phenomena (related to the loss of hyperbolicity) do not appear in this case. The slope of portion CD is always less than that of AB due to inequality $\alpha_- < \alpha_+ < 0$.

In the second case, when line $J = -\sqrt{2/3}I_1$ intersects the left part of the boundary of the undamaged states, i.e. $J_0(1 - I_1^*/I_1^+) > -\sqrt{2/3}I_1^*$, the process of damage accumulation takes place completely inside area $I_1 < I_1^*$ where the deformation due to *strong compression* plays a dominant role. The static diagram $\sigma_{11}(e)$, corresponding to this regime, has a bi-linear form

$$\sigma_{11} = \begin{cases} \Lambda_0 e, & e_M < e < 0 \\ \Lambda_f^-(e - e_M) + \sigma_M & e < e_M \end{cases},$$

where $e_M = \gamma / \alpha_-$, $\sigma_M = \Lambda_0 e_M$.

4. Head wave of damage in a tall building

Consider a problem of the head wave of damage in the initially stressed layer $0 \leq x_1 \leq H$ caused by a normal stress $\sigma_0 = -p_0 H(t)$ applied at the boundary $x^1 = 0$. Let us introduce Cartesian coordinate system with orthonormal basis \mathbf{e}_k , $k = 1, 2, 3$, such that $x^1 \equiv x$ is directed along the normal to the boundary plane formed by the other axes x^2, x^3 and $x^1 = 0$. Vector \mathbf{g} of the gravitational force is directed along x^1 . Let us assume that all material is undamaged in the initial state, all velocities are zero, and the deformation and stresses are as follows

$$\begin{aligned} \mathbf{e}^0 &= e^0 \mathbf{e}_1 \otimes \mathbf{e}_1, & \boldsymbol{\sigma}^0 &= \sigma_{11}^0 \mathbf{e}_1 \otimes \mathbf{e}_1 + \sigma_{22}^0 (\mathbf{e}_2 \otimes \mathbf{e}_2 + \mathbf{e}_3 \otimes \mathbf{e}_3) \\ \sigma_{11}^0(x) &= -\rho g x, & \sigma_{22}^0(x) &= -\lambda g x / \Lambda_0, & e^0(x) &= -g x / \Lambda_0 \end{aligned} \quad (4.1)$$

Here, λ, μ are the modules of the effective porous medium, $\Lambda_0 = \lambda + 2\mu$. The height of the building is assumed to be such that the continuous fracture does not develop in the initial state, i.e. $gH / \Lambda_0 < \gamma / |\alpha_{\pm}|$.

Normal stress $\sigma_0 = -p_0 H(t)$ is applied to the upper boundary $x=0$, where $p_0 = \text{const} > 0$ is a constant pressure and $H(t)$ is Heaviside function. Then the problem depends only on the spatial coordinate x and time t . The deformation tensor of uniaxial compression is $\mathbf{e} = e(x, t) \mathbf{e}_1 \otimes \mathbf{e}_1$, $e(x, t) < 0$. The deviator and invariants of the deformation tensor are determined by equations (3.1)

$$I_1(\mathbf{e}) = e, \quad J(\mathbf{e}) = -\sqrt{2/3}e, \quad \mathbf{e}' = e \left(\frac{2}{3} \mathbf{e}_1 \otimes \mathbf{e}_1 - \frac{1}{3} (\mathbf{e}_2 \otimes \mathbf{e}_2 + \mathbf{e}_3 \otimes \mathbf{e}_3) \right).$$

The boundary of the undamaged state of a material is approximated by a piece-wise linear function (1.10) with $\tilde{\alpha}_p(I_1) = \alpha_p^{\pm}$. Let us consider the first of the considered above regimes, i.e. the line $J = -\sqrt{2/3}I_1$ intersects the right boundary of the undamaged material. Then, we have $J_0(1 - I_1^*/I_1^+) \leq -\sqrt{2/3}I_1^*$ or

$$\alpha_p^+ I_1^+ \leq (\alpha_p^+ - \alpha_s) \sqrt{2/3} I_1^*.$$

The relationship between the stress, deformation and damage can be written in the form

$$\boldsymbol{\sigma} = \sigma_{11} \mathbf{e}_1 \otimes \mathbf{e}_1 + \sigma_{22} (\mathbf{e}_2 \otimes \mathbf{e}_2 + \mathbf{e}_3 \otimes \mathbf{e}_3), \quad \sigma_{11} = \Lambda_0 e - \alpha_{\pm} \omega, \quad \sigma_{22} = \lambda e - \hat{\alpha}_{\pm} \omega, \quad (4.2)$$

where

$$\alpha_{\pm} \equiv \alpha_p^{\pm} - \sqrt{2/3} \alpha_s, \quad \hat{\alpha}_{\pm} \equiv \alpha_p^{\pm} + \sqrt{1/6} \alpha_s. \quad (4.3)$$

Signs plus and minus of symbols $\alpha_p^{\pm}, \alpha_{\pm}, \hat{\alpha}_{\pm}$ correspond to intervals $I_1^* < I_1 < 0$ and $I_1 < I_1^*$, respectively (i.e. to the portions AB and CD of function $\sigma_{11}(e)$ shown in Fig. 4).

The uniaxial dynamic deformation of the considered medium is described the system of equations comprising motion equation (1.5), kinetic equation (3.5), and equation (1.8) for the stress tensor. This system can be reduced to three partial-differential equations with respect to the solution vector $(v_1, \sigma_{11}, \omega)$

$$\rho \frac{\partial v_1}{\partial t} - \frac{\partial \sigma_{11}}{\partial x} = \rho g, \quad \frac{\partial \sigma_{11}}{\partial t} - \Lambda_0 \frac{\partial v_1}{\partial x} = -\frac{\alpha_{\pm}}{\tau} z^{\pm}(\sigma_{11}, \omega), \quad \frac{\partial \omega}{\partial t} = \frac{1}{\tau} z^{\pm}(\sigma_{11}, \omega). \quad (4.4)$$

For convenience, let us introduce non-dimensional variables

$$\begin{aligned} \bar{t} &= \frac{t}{t_0}, & \bar{x} &= \frac{x}{H}, & \bar{v} &= \frac{v_1}{c_0}, & \bar{\sigma} &= \frac{\sigma_{11}}{\Lambda_0}, & \bar{\sigma}_0 &= \frac{\sigma_0}{\Lambda_0}, & \bar{g} &= \frac{\rho g H}{\Lambda_0}, \\ \frac{1}{\bar{\tau}} &= \frac{t_0}{\tau}, & \bar{\lambda} &= \frac{\lambda}{\Lambda_0}, & \bar{\mu} &= \frac{\mu}{\Lambda_0}, & \bar{\alpha}_{\pm} &= \frac{\alpha_{\pm}}{\Lambda_0}, & \bar{\beta} &= \frac{\beta}{\Lambda_0}, & \bar{\gamma} &= \frac{\gamma}{\Lambda_0} \end{aligned} \quad (4.5)$$

where $c_0 = (\Lambda_0 / \rho)^{1/2}$ is the velocity of elastic longitudinal waves, $t_0 = H / c_0$ is the characteristic time, and $\bar{\lambda} + 2\bar{\mu} = 1$. The system (4.4) written in non-dimensional variables will look in this form as

$$\frac{\partial v}{\partial t} - \frac{\partial \sigma}{\partial x} = g, \quad \frac{\partial \sigma}{\partial t} - \frac{\partial v}{\partial x} = -\frac{\alpha_{\pm}}{\tau} z^{\pm}, \quad \frac{\partial \omega}{\partial t} = \frac{1}{\tau} z^{\pm}, \quad (4.6)$$

$$z^{\pm}(\sigma, \omega) \equiv (\alpha_{\pm} \sigma - \gamma) / \beta - b_{\pm}^2 \omega, \quad b_{\pm}^2 = 1 - \alpha_{\pm}^2 / \beta.$$

Here and below, the overbar is omitted for brevity. The boundary and initial conditions for the system (4.6) are the following

$$\sigma(x, 0) = -gx, \quad v(x, 0) = \omega(x, 0) = 0, \quad 0 \leq x \leq 1, \quad (4.7)$$

$$\sigma(0, t) = \sigma_0, \quad v(1, t) = 0, \quad t \geq 0. \quad (4.8)$$

The system (4.6) is a quasi-linear hyperbolic system of equations with a linear differential part. The characteristics velocity is determined by the equation

$$\det \begin{bmatrix} -c & -1 & 0 \\ -1 & -c & 0 \\ 0 & 0 & -c \end{bmatrix} = 0 \text{ or } c(c^2 - 1) = 0.$$

It follows that the non-stationary characteristics propagate with speed $c = \pm 1$ in the considered material. Because of the linearity of a differential part of the system, the strong shocks propagate with the same velocity.

Due to the discontinuity of the boundary and initial conditions (4.7) and (4.8) at $x=0, t=0$, the strong shock appears in the solution originating from $x=0, t=0$. Therefore, on the interval $0 \leq t \leq 1$ (corresponding to the first run of the wave along the building) the system (4.6) can be considered in the triangle area $\{0 \leq x \leq t, t \geq 0\}$, where $x(t) = t$ is the position of the shock moving with velocity $c=1$ through the initially stressed medium. As the boundary conditions, the first equation of (4.8) will be used

$$\sigma(0, t) = \sigma_0, \quad t \geq 0, \quad (4.9)$$

as well as the relations for the strong shock at $x=t$:

$$[\sigma] + [v] = 0, \quad [\omega] = 0.$$

By accounting for equations (4.7), the jump conditions can be written as follows

$$v(t, t) = -(gx(t) + \sigma(t, t)), \quad \omega(t, t) = 0. \quad (4.10)$$

Below we consider the evolution of the stress $\sigma_*(t) \equiv \sigma(t-0, t)$ at the head shock. In this process, three main possible variants can be analyzed, corresponding to the high, moderate, and low pressure applied to the boundary $x=0$.

At high stress $-p_0 \equiv \sigma_0 < \sigma_B \equiv I_1^* = e_B$, the shock compression of the initially porous material causes the volumetric deformation whose absolute value exceeds $|I_1^*|$, which separates the regimes of shear and volumetric destruction. In this case, the minus sign should be selected in coefficients α_{\pm} corresponding to deformation $e < I_1^*$. The right-hand side of the kinetic equation at the point $x=t=0$ is greater than zero; therefore, the volumetric damage starts immediately. The propagation of the shock is accompanied by the reduction of compressing stress $\sigma_*(t)$. At time moment t_B (when $\sigma_*(t_B) = e_B$), the volumetric character of damage transforms into shear damage. The stress continues to decrease (in absolute value) approaching to the limiting value $\sigma_{\infty} < \sigma_A = \gamma / \alpha_+$. The stress evolution immediately behind the shock is determined by the solution of differential equations with initial conditions

$$\begin{aligned} \frac{d\sigma_*}{dt} &= -\frac{\alpha_-^2}{2\tau\beta} (\sigma - \gamma / \alpha_-) - g, \quad \sigma_*(0) = -p, \quad 0 < t < t_B \\ \frac{d\sigma_*}{dt} &= -\frac{\alpha_+^2}{2\tau\beta} (\sigma - \gamma / \alpha_+) - g, \quad \sigma_*(t_B) = \sigma_B, \quad t_B < t \end{aligned} \quad (4.11)$$

The solution of Cauchy problem (4.11) is the implicit dependence

$$t = - \int_{\sigma_0}^{\sigma_*} \left(\frac{\alpha_-^2}{2\tau\beta} (\sigma - \gamma / \alpha_-) + g \right)^{-1} d\sigma, \quad 0 < t < t_B \quad (4.12)$$

$$t - t_B = - \int_{\sigma_B}^{\sigma_*} \left(\frac{\alpha_+^2}{2\tau\beta} (\sigma - \gamma / \alpha_+) + g \right)^{-1} d\sigma, \quad t_B < t$$

The second equation (4.11) shows that if the relaxation time is small $\tau \ll 1$, then stress σ_* approaches its asymptotic value σ_∞ , which is defined by the equation for zero time derivative of stress:

$$\frac{\alpha_+^2 (\sigma_\infty - \sigma_A)}{2\tau\beta} + g = 0$$

or

$$\sigma_\infty = \frac{\gamma}{\alpha_+} - \frac{2\tau\beta g}{\alpha_+^2} = \sigma_A - \frac{2\tau\beta g}{\alpha_+^2} = \sigma_A - \frac{2\tau g}{1 - b_+^2}. \quad (4.13)$$

The threshold value σ_A differs from the value σ_∞ , which is caused by the combined action of kinetics and initial compression of the material. This effect is absent in wave propagation inside elastic-viscoplastic media (Kukudjanov 1977) and in damaged media without initial stresses (Kondaurov 1998, Kondaurov & Kutlyarova 2000), where the asymptotic value is exactly equal to the threshold value σ_A .

Equations (4.12) can be written in the explicit form

$$\sigma_*(t) - \sigma_0 = \left(\frac{\alpha_+}{\alpha_-} \sigma_A - \sigma_0 - \frac{2\tau g}{1 - b_-^2} \right) \left[1 - \exp \left(- \frac{\alpha_-^2}{\alpha_+^2} \frac{gt}{(\sigma_A - \sigma_\infty)} \right) \right], \quad \sigma_0 \leq \sigma_*(t) \leq \sigma_B \quad (4.14)$$

$$\sigma_*(t) - \sigma_B = (\sigma_\infty - \sigma_B) \left[1 - \exp \left(- \frac{g(t - t_B)}{\sigma_A - \sigma_\infty} \right) \right], \quad \sigma_B \leq \sigma_*(t) < \sigma_\infty$$

From these equations, it follows that the compression stress decreases exponentially in the head wave of damage. Moreover, the exponential coefficient of attenuation for the shear mechanism of damage (the second equation), is equal to the ratio of the velocity of free fall $g(t - t_B)$ to the width of the *kinetic split*, which is

$$\Delta_{kin} \equiv \sigma_A - \sigma_\infty = 2\tau g / (1 - b_+^2).$$

In the volumetric damage (the first equation), the attenuation coefficient is greater because $\alpha_-^2 / \alpha_+^2 > 1$.

Figure 5 presents a dependence of the amplitude $\sigma_*(t) / \sigma_A$ in the head wave of the material with parameters $\lambda = \mu = 0.333$, $\alpha_p^+ = 0.3$, $\alpha_p^- = -0.1$, $\gamma = 0.01$, $\beta = 1.0$, $\alpha_s = 1.1$, $\sigma_0 = 10\sigma_A$, $e_B = 3e_A$ for three relaxation times $\tau = 0.01, 0.03, 0.05$ (curves 1-3). The break in slope is noticeable that occurs at time

$$t_B = - \frac{\alpha_+^2 (\sigma_A - \sigma_\infty)}{\alpha_-^2 g} \ln \left[1 - \frac{\sigma_B - \sigma_0}{\frac{\alpha_+}{\alpha_-} \sigma_A - \sigma_0 - \frac{2\tau g}{1 - b_-^2}} \right],$$

which corresponds to the transition from the volumetric to the shear damage mechanism in the head wave.

In the case of *moderate stress* $\sigma_B < \sigma_0 < \sigma_A$, applied to the upper boundary $x=0$, the damage starts also at $t=0$. The shear mechanism of damage is realized. The stress evolution on the shock is determined by one differential equations and one initial condition

$$\frac{d\sigma_*}{dt} = - \frac{\alpha_+^2}{2\tau\beta} (\sigma - \sigma_A) - g, \quad \sigma_*(0) = -p, \quad t \geq 0. \quad (4.15)$$

The solution for this problem is

$$\sigma_*(t) - \sigma_0 = (\sigma_\infty - \sigma_0) \left[1 - \exp \left(- \frac{gt}{\sigma_A - \sigma_\infty} \right) \right]. \quad (4.16)$$

Let stress σ_0 (applied to the boundary) be in the range $\sigma_B < \sigma_0 < \sigma_\infty$, i.e. below limiting value σ_∞ . Then coefficient $\sigma_\infty - \sigma_0$ before the square brackets in (4.16) is positive. Amplitude $|\sigma_*(t)|$ of the head wave decreases monotonically from value $|\sigma_0|$ to the limit $|\sigma_\infty|$ defined by equation (4.13). If the stress is within the kinetic split, i.e. $\sigma_\infty < \sigma_0 < \sigma_A$, then amplitude $|\sigma_*(t)|$ increases up to value $|\sigma_\infty|$.

Figure 6 presents the dependence of amplitude $\sigma_*(t)/|\sigma_A|$ of the head wave caused by moderate pressure $\sigma_0=2\sigma_A$ (curves 1,3,5) and $\sigma_0=2.9\sigma_A$ (curves 2,4,6) for parameters $\lambda = \mu = 0.333$, $\alpha_p^+ = 0.3$, $\alpha_p^- = -0.1$, $\gamma = 0.01$, $\beta = 1.0$, $\alpha_s = 1.1$, $e_B = 3e_A$ for three relaxation times: $\tau = 0.01$ (curves 1,2), $\tau = 0.03$ (curves 3,4), and $\tau = 0.05$ (curves 5,6). Depending of the relaxation time τ , the given load σ_0 can lead to principally different behavior of the head wave amplitude in time. The amplitude may decrease (curves 1,2,4), increase (curves 3,5), or remain the same (curve 6) in comparison with initial value σ_0 . The increase of the amplitude is the effect related to the presence of initial stresses.

The third regime appears when stress σ_0 applied to the boundary $x=0$ is so small that $z^+(\sigma_0, 0) \equiv \alpha_+ \sigma_0 - \gamma \leq 0$. No damage is present in the initial phase of the shock propagation, because the right-hand side of the damage kinetic equation $\tau \dot{\omega} = z^+$ is zero for $z^+ \leq 0$. The solution for the system (4.6), (4.9)-(4.10) at $z^+ \leq 0$ is piece-wise constant

$$\begin{aligned} \sigma(x,t) &= \sigma_0 - gx, & v(x,t) &= -\sigma_0, & \omega(x,t) &= 0, & 0 \leq x \leq t, t > 0 \\ \sigma(x,t) &= -gx, & v(x,t) &= 0, & \omega(x,t) &= 0, & t < x, t > 0 \end{aligned} \quad (4.17)$$

The maximum absolute value of the compressing stress $\sigma(1,1) = \sigma_0 - gx$ is achieved at the base of the layer $x=1$ at $t=1$. The material in this section will be undamaged if $\sigma_0 \geq \sigma_0^e \equiv g + \gamma / \alpha_+$.

If the stress $p = -\sigma_0$ applied at the boundary $x=0$ exceeds $p^e = -\sigma_0^e$, then from equation (4.17) it follows that the acting stress attains threshold value $\sigma_A = \gamma / \alpha_+$ at point $(x_A, t_A = x_A)$, where the following equality is held

$$\sigma_0 = gx_A + \gamma / \alpha_+, \quad (4.18)$$

and where the process of damage accumulation starts, i.e. the elastic precursor transforms into the damage wave. Behind the shock, stress $\sigma_*(t)$, which prior to that moment was exactly equal to the sum of the stress applied at the boundary and the initial stress, is now determined by formula

$$\sigma_*(t) - \sigma_A = -(\sigma_A - \sigma_\infty) \left[1 - \exp\left(-\frac{g(t-t_A)}{\sigma_A - \sigma_\infty}\right) \right]. \quad (4.19)$$

From this equation, it follows that stress $\sigma_*(t) \rightarrow \sigma_\infty$ as t increases.

Figure 7 shows the solution for two values of stress $\sigma_0=0.3\sigma_A$ (curves 1,2) and $\sigma_0=0.9\sigma_A$ (curves 3,4,5) for the material with parameters given above for three relaxations times $\tau = 0.001$ (curve 5), $\tau = 0.01$ (curves 2,4), and $\tau = 0.02$ (curves 1,3). The solution demonstrates that at low loads, the volumetric damage does not appear in the head shock, and damage is caused only by the shear mechanism. Unlike to interval $t \leq t_A$, where the stress rate is $d\sigma/dt = -g$, at subsequent times $t > t_A$ the rate is

$$\frac{d\sigma}{dt} = -g - \frac{1-b_+^2}{2\tau}(\sigma - \sigma_A) > -g,$$

i.e. the negative slope of curve $\sigma_*(t)$ decreases. This means that the material with initial stresses unloads in the vicinity of the head shock $x=t$, remaining compressed below threshold value σ_A .

5. Building collapse

The constructed solution together with the condition (2.12) of the reological instability allows us to determine how the macro-failure wave propagates in tall buildings. On the head wave, which corresponds to the line $\eta = 0$ in characteristic variables $\zeta = \frac{1}{2}(x+t)$, $\eta = \frac{1}{2}(t-x)$, the damage is absent

$$\omega(\zeta, 0) = 0, \quad (5.1)$$

and the stress is found from equations (4.14), (4.16), and (4.19). For constructing the solution at $\eta > 0$, derivatives $\partial v / \partial \eta$, $\partial \sigma / \partial \eta$, $\partial \omega / \partial \eta$ must be known. Taking into account $t = \zeta + \eta$ and

$\partial \omega / \partial t|_{\zeta=const} = \partial \omega / \partial \eta$, from the third equation (4.6) it follows

$$\frac{\partial \omega}{\partial \eta} = \frac{1}{\tau} z^{\pm}(\zeta, \eta), \quad z^{\pm} \equiv \frac{\alpha_{\pm} \sigma - \gamma}{\beta} - \left(1 - \frac{\alpha_{\pm}^2}{\beta}\right) \omega = \frac{\alpha_{\pm}}{\beta} \left(\sigma - \frac{\gamma}{\alpha_{\pm}}\right) - b_{\pm}^2 \omega,$$

where $b_{\pm}^2 = (1 - \alpha_{\pm}^2 / \beta)$. Therefore, in the vicinity of characteristics $\eta = 0$ one can obtain

$$\omega(\zeta, \eta) = \eta \tau^{-1} z_{*}^{\pm}(\zeta), \quad (5.2)$$

where

$$z_{*}^{\pm}(\zeta) \equiv z^{\pm}(\zeta, 0) = (\alpha_{\pm} \sigma_{*}(\zeta) - \gamma) / \beta, \quad (5.3)$$

which is a semi-positive function that has a discontinuity at $t = t_B$ due to a jump of α .

Derivative $\partial \sigma / \partial \eta$ in the same vicinity satisfies the first-order equation

$$\frac{dY_{\pm}}{d\zeta} + A_{\pm} Y_{\pm} = \frac{\alpha_{\pm} b_{\pm}^2}{2\tau^2} z_{*}^{\pm}(\zeta) - \frac{\alpha_{\pm}}{2\tau} \frac{dz_{*}^{\pm}(\zeta)}{d\zeta}, \quad Y_{\pm}(\zeta) = \frac{\partial \sigma}{\partial \eta} \Big|_0, \quad A_{\pm} = \frac{\alpha_{\pm}^2}{2\tau\beta}, \quad (5.4)$$

with the initial condition at $\zeta = 0$

$$Y_{\pm}(0) = g + A_{\pm} \left(\sigma_0 - \frac{\gamma}{\alpha_{\pm}}\right) = g + \frac{\alpha_{\pm}}{2\tau} z_{*}^{\pm}(0). \quad (5.5)$$

Sign plus corresponds to point ζ such that $\sigma_B < \sigma_{*}(\zeta) < \sigma_A$, and minus to the points ζ where $\sigma_{*}(\zeta) < \sigma_B$. The solution of Cauchy problem (5.4)-(5.5) for the case of the moderate applied stress $\sigma_B < \sigma_0 < \sigma_A$ is

$$Y_{+}(\zeta) = \left(g + \frac{\alpha_{+}}{\tau} z_{*}^{+}(0)\right) e^{-A_{+}\zeta} - \frac{\alpha_{+}}{2\tau} z_{*}^{+}(\zeta) + B_{+} e^{-A_{+}\zeta} \int_0^{\zeta} z_{*}^{+}(s) e^{A_{+}s} ds, \quad (5.6)$$

and for high stress $\sigma_0 < \sigma_B$ it is

$$\begin{aligned} Y_{-}(\zeta) &= \left(g + \frac{\alpha_{-}}{\tau} z_{*}^{-}(0)\right) e^{-A_{-}\zeta} - \frac{\alpha_{-}}{2\tau} z_{*}^{-}(\zeta) + B_{-} e^{-A_{-}\zeta} \int_0^{\zeta} z_{*}^{-}(s) e^{A_{-}s} ds, & \zeta < \zeta_B \\ Y_{+}(\zeta) &= \left[Y_{-}(\zeta_B) + \frac{\alpha_{+}}{2\tau} z_{*}^{+}(\zeta_B)\right] e^{-A_{+}(\zeta - \zeta_B)} - \frac{\alpha_{+}}{2\tau} z_{*}^{+}(\zeta) + B_{+} e^{-A_{+}\zeta} \int_{\zeta_B}^{\zeta} z_{*}^{+}(s) e^{A_{+}s} ds, & \zeta > \zeta_B \end{aligned} \quad (5.7)$$

Here, we introduced

$$B_{\pm} = \frac{\alpha_{\pm}}{\tau^2} \left(1 - \frac{\alpha_{\pm}^2}{2\beta}\right) = \frac{\alpha_{\pm}(1 + b_{\pm}^2)}{4\tau^2}, \quad b_{\pm}^2 = 1 - \frac{\alpha_{\pm}^2}{\beta},$$

and

$$\zeta_B = -\frac{\alpha_{+}^2 (\sigma_A - \sigma_{\infty})}{\alpha_{-}^2 g} \ln \left[1 - \frac{\sigma_B - \sigma_0}{\alpha_{+} \sigma_A / \alpha_{-} - \sigma_0 - 2\tau g / (1 - b_{-}^2)}\right].$$

The value ζ_B is the time moment of transition from the volumetric to shear damage mechanism.

Now everything is ready for investigating the macro-failure wave. Let us use condition (2.12) of the onset of reological instability $\alpha_s \omega(\zeta, \eta) = \xi_{cr} J(\zeta, \eta)$. Accounting for equations (3.1) and (3.4), which gives

$$J(\zeta, \eta) = -\sqrt{\frac{2}{3}} (\sigma(\zeta, \eta) + \alpha_{\pm} \omega(\zeta, \eta)), \text{ and using equation (5.2) and relation}$$

$\sigma(\zeta, \eta) = \sigma_*(\zeta) + \eta \partial\sigma / \partial\eta|_0 + O(\eta^2)$, where derivative $\partial\sigma / \partial\eta|_0$ is determined by equations (5.4) and (5.5), the equation for coordinate $\eta_*^\pm(\zeta)$ of the macro-destructive wave is

$$\eta_*^\pm(\zeta) = \frac{|\sigma_*(\zeta)|}{F^\pm(\zeta)}, \quad F^\pm(\zeta) = \frac{\sqrt{3/2}\alpha_s / \xi_{cr} - |\alpha_\pm|}{\tau} z_*^\pm(\zeta) + \frac{\partial\sigma(\zeta, 0)}{\partial\eta}, \quad (5.8)$$

where the amplitude $\sigma_*(\zeta)$ of the head wave is given by one of expressions (4.14), (4.16), or (4.19) and positive function $z_*^\pm(\zeta)$ is determined by equation (5.3). In variables (x, t) , equation (5.8) can be written as

$$x(t) = t - 2 \frac{|\sigma_*(t)|}{F^\pm(t)}, \quad F^\pm(t) = \frac{\sqrt{3/2}\alpha_s / \xi_{cr} - |\alpha_\pm|}{\tau} z_*^\pm(t) + Y_\pm(t). \quad (5.9)$$

If the macro-failure wave is realized, than its coordinate $\eta_*^\pm(\zeta) > 0$. This is possible only at a positive value of $F^\pm(\zeta)$. Since parameter ξ_{cr} enters only the first term of $F^\pm(\zeta)$, function $F^\pm(\zeta)$ is maximal at the lowest value $\xi_{cr} = 2\mu$ for any fixed parameter ζ . Therefore, if the solution exists, then the fastest wave of macro-damage is characterized by minimal value $\eta_*(\zeta)$ and by the first form of reological instability. This means that macro-damage in the form of cones (3.8) or planes (3.6), which are inclined with respect to the vertical axis of a building, does not realize. Figure 8a shows coordinate $x = x_c(t)$ of the damage wave as a function of time for three values of moderate load $\sigma_0 > \sigma_B$. Curve 1 corresponds to line $x = x_c(t)$, along which the head wave propagates, and curves 2-4 are given for the lines at loads $\sigma_0 / \sigma_A = 2.3, 2.4, \text{ and } 2.8$. The following parameters were used $\lambda = \mu = 0.333, \alpha_p^+ = 0.3, \alpha_p^- = -0.1, \gamma = 0.01, \beta = 1.0, \alpha_s = 1.1, e_B = 3e_A, g = 0.1, \tau = 0.02$.

The non-trivial influence of the relaxation time on the appearance of the macro-failure wave should be emphasized. Figure 8b gives dependence of $x = x_c(t)$ for the same material parameters but for $\tau = 0.01$. In this case, the damage wave appears at loads $\sigma_0 / \sigma_A > 2.6$, while for $\tau = 0.02$ the macro-failure wave started at $\sigma_0 / \sigma_A > 2.25$. This means the strength of the building increases if the material with faster kinetics is used. The increase of the shear modulus also leads to the strength augmentation. For $\mu = 0.45, \lambda = 0.10, \tau = 0.01$, the shear macro-damage starts at $\sigma_0 / \sigma_A > 2.66$.

An interesting phenomenon is observed in the case of intensive dynamic load $\sigma_0 < \sigma_B$. For parameters given above, the time dependence of the macro-failure wave coordinate $x = x_c(t)$ depends only weakly on the applied load and practically coincides (with accuracy of a few percent) with the dependence for load $\sigma_0 = \sigma_B$ equal to the stress at which the volumetric destruction of the material starts. This is related to the fast attenuation of the wave in the vicinity of boundary $x=0$ up to the value $\sigma_\infty = \sigma_A - 2\tau\beta g / \alpha_+^2$. The significance of this stress was described in Section 4.

Figure 9 presents isolines of critical parameter

$$\xi_{cr}(\zeta, \eta) \equiv 2\mu - \frac{\alpha_s \omega}{J} = 2\mu + \frac{\sqrt{1.5}\alpha_s \eta z(\zeta)}{\alpha \eta z(\zeta) + \tau (\sigma_*(\zeta) + \eta Y(\zeta))}$$

on the plane (x, t) for loads $\sigma_0 = \sigma_B + 0.1\sigma_A$ (Fig. 9a) and $\sigma_0 = \sigma_B + 17\sigma_A$ (Fig. 9b). Line $\xi_{cr}(\zeta, \eta) = 0$, which corresponds to the macro-damage wave, varies little in a broad range of dynamic load amplitudes.

Conclusions

A relatively simple model is proposed for describing the damage accumulation (scattered destruction) in materials with high initial porosity. This approach can be applied to model several specific features of deformation and damage in tall buildings. These features include:

- Threshold values of compressive deformation for the onset of the continuous fracture.
- Reduction of stiffness of the effective medium due to damage accumulation.

- Strong dependence of the stress and damage evolution on the form of the elastic region of material.
- Instability phenomena that are related to the macro-fracture and characterize the strength under tension, shear, and compression of a porous medium. These limiting states depend not only on the material properties but also on the strain history.

The propagation of the head wave is investigated, which is caused by a step-compressing load applied to the upper boundary of a building. It is shown that a relatively weak impulse may lead to the appearance of the damage wave due to the presence of initial stresses in structural elements in the gravity field.

It is also proved that moderate loads result only in shear damage in the head wave. Depending on the level of the applied load, stresses may either increase or decrease in the process of the head wave propagation.

It is found that intensive loads may lead to the volumetric destruction, which realizes only in the first phase of motion. Although this wave travels into the region with increasing initial stresses, it inevitably transforms into the wave of shear damage.

It is shown that the stress evolution in shear damage wave is characterized by exponential attenuation with a coefficient equal to the ratio of velocity in free fall to the width of a so-called kinetic split. In the wave of volumetric damage, this attenuation coefficient is greater than in the shear wave.

A new effect of the existence of the limiting stress is noticed, which is different from the threshold stress level, corresponding to the onset of damage accumulation. The appearance of this limiting stress is caused by combined influence of damage kinetics and initial stresses.

The equation for the propagation of a damage wave is obtained. The conditions for the appearance of this wave and the dependence of time delay in the building collapse on the applied dynamic load are studied.

It is shown that for preventing the onset of reological instability, which is recognized as macroscopic destruction of material, one needs to increase shear modulus μ and coefficient β that characterizes nonlinear dissipation of released energy, which transforms into the effective surface energy. Absolute values of coefficients α_s , α_{\pm} of the effective porous medium must be as low as possible, because they characterize the energy release in damage accumulation.

A new effect is found related to the weak dependence of the damage wave motion on the value of the applied load exceeding threshold value σ_B , when the shear damage transforms into volumetric damage.

Acknowledgement

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Literature

- Aubertin, M., Li, L., Simon, R. 2000 A multiaxial stress criterion for short- and long-term strength of isotropic rock media. *International Journal of Rock Mechanics and Mining Sciences*, 37, 1169-1193.
- Bahvalov, N.S. 1974 Averaged characteristics of the bodies with periodic structure. *Doklady Akademii Nauk USSR*. 218(5), 1046-1048.
- Bahvalov, N.S. 1975 Averaging of differential operators with partial derivatives with fast oscillating coefficients. *Doklady Akademii Nauk USSR*. 221(3), 516-519.
- Bazant, Z.P., Zhou, Y. 2001 Why did World Trade Center collapse? – Simple analysis. *Archive of Applied Mechanics*. 71(12), 802-806.
- Berdichevsky, V.L. 1983 Variational principles of mechanics of continuous medium. Nauka, Moscow.
- Cherepanov, G.P. 1974 Mechanics of Brittle Fracture. Nauka, Moscow.
- Christensen, R.M. 1979 Mechanics of Composite Materials, Wiley-Inter-Science, New York.
- Courant, R. 1962 Partial Differential Equations, Interscience, New York.
- Fleck, N.A., Kuhn, L.T., McMeeking, R.M. 1992 Yielding of metal powder bonded by isolated contacts. *Journal of the Mechanics and Physics of Solids*. 40(5), 1139-1162.
- Griffith, A.A. 1920 The phenomena of rupture and flow in solids. *Philosophical Transactions of the Royal Society A*. 221, 163-198.
- Holzappel, G.A. 2000 Nonlinear Solid Mechanics: A Continuum Approach for Engineering. John Wiley & Sons.
- Freund, L.B. 1998 Dynamic fracture mechanics. Cambridge University Press, New York.
- Kondaurov, V.I. 1988 Continual fracture of nonlinear-elastic bodies. *Applied Mathematics and Mechanics*. 52(2), 302-310.
- Kondaurov, V.I. 1991 On the rheological instability of elastic damaged medium. *Applied Mathematics and Mechanics*. 55(1), 109-117.
- Kondaurov, V.I. 1998 On the failure waves features in highly homogenous brittle materials. *Applied Mathematics and Mechanics*. 62(4), 722-729.
- Kondaurov, V.I., Kutlyarova, N.V. 2000 Damage and rheological instability of initially porous materials. *Izvestia Akademii Nauk USSR. Mechanics of Solids*. № 4, 99-109.
- Kondaurov, V.I., Fortov, V.E. 2002 Fundamentals of Thermomechanics of Condensed Matter. Moscow Institute of Physics and Technology, Moscow.
- Kukudjanov, V.N. 1977 On wave propagation in a coupled thermo-elastic-plastic medium. *Archive of Applied Mechanics*. 29(2). 325-338.
- Lundgren, K., Magnusson J., 2001 Three-dimensional modeling of anchorage zones in reinforced concrete. *Journal of Engineering Mechanics*, 7, 693-699.
- Ning, L., Zhang, P., Qingfwi, D., Swoboda, G. 2003 Dynamic damage model of the rock mass medium with microjoints. *International Journal of Damage Mechanics*, 12, 163-173.
- Rudnicki, J.W., Rice J.R. 1975 Conditions for localization of deformation in pressure-sensitive dilatant materials. *Journal of the Mechanics and Physics of Solids*. 23. P. 371 – 393.
- Truesdell, C. 1972 A first course in rational continuum mechanics The John Hopkins University. Baltimore. Maryland
- Zhang, Y.Q., Hao, H. 2003 Dynamic fracture in brittle solids at high rates of loading. *Journal of Applied Mechanics*. 70, 454-457.

Figure captions

Fig. 1. Boundary between intact and damaged material.

Fig. 2. Strength boundaries (adapted from Aubertin & Simon 2000).

Fig. 3. Examples of piece-wise linear approximations for the boundary of intact material area (solid and dash-dotted lines). Other lines and symbols are referred to in Section 4.

Fig. 4. Quasi-static diagram for the stress-strain relationship in the first regime.

Fig. 5. Stress amplitude in the head wave for high applied stress.

Fig. 6. Stress amplitude in the head wave for moderate applied stress.

Fig. 7. Stress amplitude in the head wave for low applied stress.

Fig. 8. Coordinate of the damage wave for various loads.

Fig. 9. Isolines of the critical parameter for two applied loads.

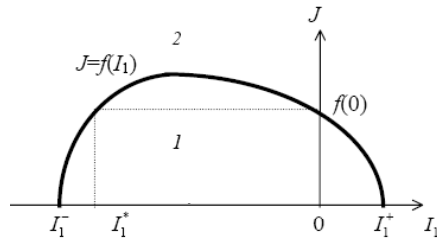


Fig. 1

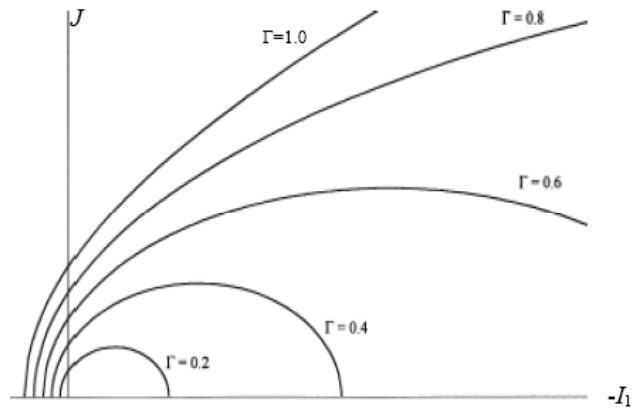


Fig. 2

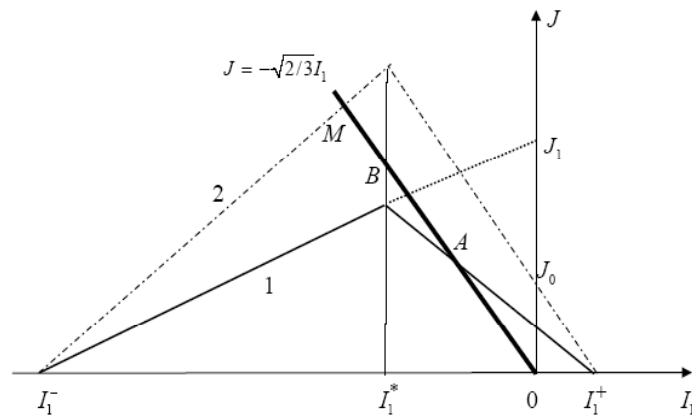


Fig. 3

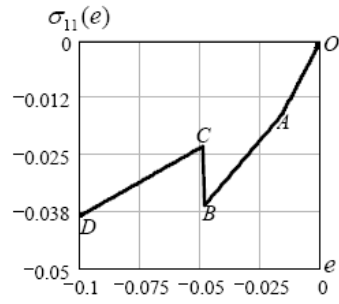


Fig. 4

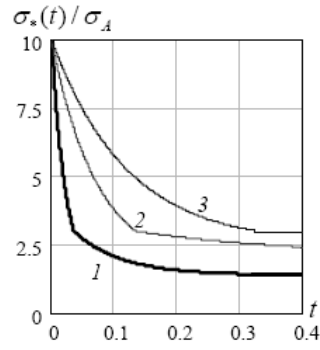


Fig. 5

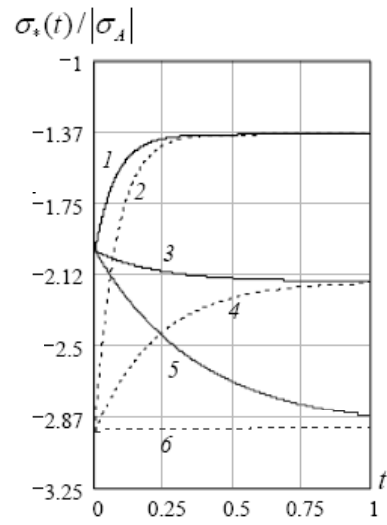


Fig. 6

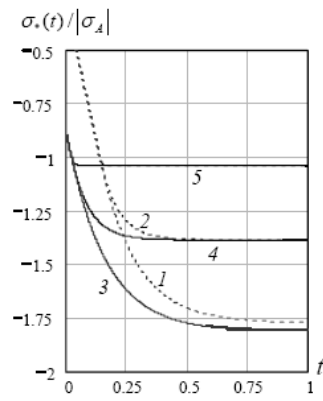


Fig. 7

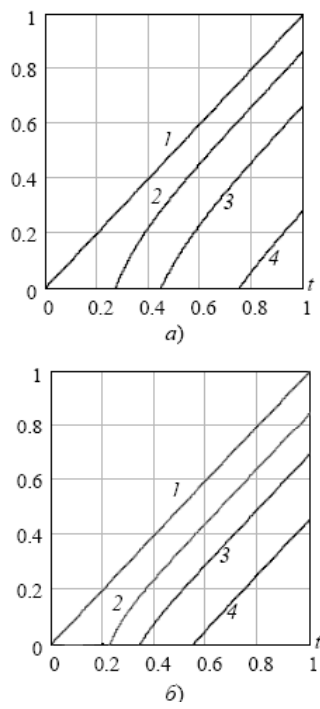


Fig. 8

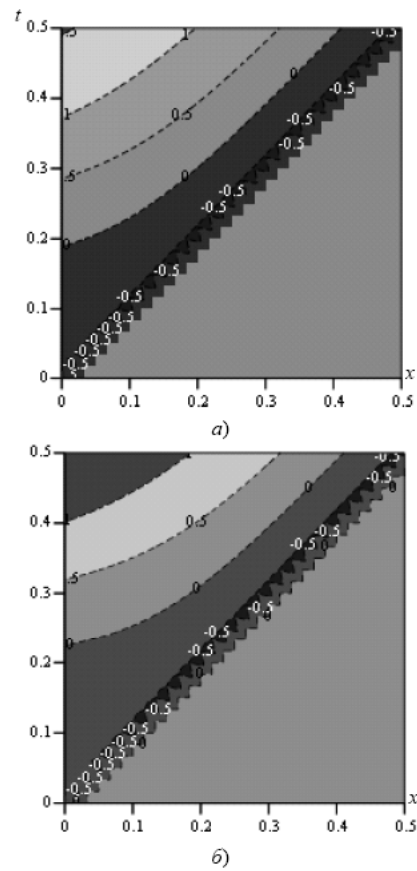


Fig. 9